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On Global Regular Solutions of Third Order Partial Differential Equations

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INTRODUCTION

The aim of this paper is the study of initial value problems for third order partial differential equations of the following type

$$\frac{\partial^2 u}{\partial t^2} + A\frac{\partial u}{\partial t} + G\left(u, \nabla u, \nabla^2 u, \frac{\partial u}{\partial t}, \nabla\frac{\partial u}{\partial t}\right) = 0,$$

$$u(x, 0) = \varphi(x), \qquad \frac{\partial u}{\partial t}(x, 0) = \psi(x),$$

where A denotes a second order elliptic differential operator with smooth and bounded coefficients. We are interested in unique global smooth solutions, i.e. solutions defined in $\mathbb{R}^n \times (0, \infty)$ having derivatives in the classical sense up to the order required by the equation. Equations of this type are related to parabolic problems of second order by setting $\partial u/\partial t = v$. We consider two different cases.

In the first section there is treated the differential equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta\frac{\partial u}{\partial t} - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\sigma_i\left(\frac{\partial u}{\partial x_i}\right)\right) = 0,$$

where Δ denotes the spatial Laplacian. The existence of a unique solution locally in time can be shown for arbitrary dimensions by means of Banach's fixed point theorem applied to a corresponding integral equation. The needed a-priori-estimates for nonlinearities being *not* sublinear can be established in dimension $n \leqslant 2$, if σ_i is monotonically increasing and for $n = 2$ of at most cubic growth. Related initial-boundary value problems in bounded domains and *one* space dimension were discussed in [3], [5], and [6].

In the second part of this paper we consider equations of the form $\partial^2 u/\partial t^2 + A(\partial u/\partial t) + f_1(u, \nabla u, u_t, \nabla u_t) + f_2(u) + \sum_{i=1}^{n} (g_i'(u)\, u_{x_i} u_t + 2g_i(u)\, u_{x_i t}) = 0,$

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where the nonlinear terms satisfy certain growth conditions and moreover essentially the sign conditions $f_1 \cdot u_t \geqslant 0$, and f_2 has a nonnegative primitive function. The first a-priori-estimate, the so-called energy estimate, is easily done by means of scalar multiplication of the equation by u_t. The next step, an estimate of $\| u(t)\|_{2-\epsilon,n/2}$, requires a rather lengthy calculation (Lemma 2.3) using some calculus lemmas of Sobolev type which allow suitable estimates for the nonlinear term in the corresponding integral equation (of parabolic type) for $u_t(t)$. Having done this it is not very hard to estimate any desired norm of the solution. Some special kinds of this equation in $n = 1$ were already treated in [1].

Let us introduce some notation. $\mathbb{R}^+$ denotes the half-axis $\{t \in \mathbb{R} \mid t \geqslant 0\}$, $D^\alpha = \partial^{\alpha_1}/\partial x_1^{\alpha_1} \cdots \partial^{\alpha_n}/\partial x_n^{\alpha_n}$ for $\alpha = (\alpha_1, ..., \alpha_n) \in (\mathbb{N} \cup \{0\})^n$. The following function spaces appear: $C^k(A, X)$, $k \in \mathbb{N} \cup \{0\}$, X Banach space, $A \subset \mathbb{R}^n$ closed set, denotes the Banach space of k-times continuously differentiable mappings $u: A \to X$. $C^k_{\mathrm{loc}}(G, C)$ is the set of k-times continuously differentiable mappings $u: G \to C$ for $G \subset \mathbb{R}^n$, $C \subset \mathbb{R}$. For $S \in \mathbb{R}^+$, $1 < p < \infty$, let $H^{S,p}(\mathbb{R}^n) = H^{S,p}$ be the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{R})$ with respect to the norm $\|f\|_{S,p} = \| \mathscr{F}^{-1}[(1 + |\xi|^2)^{S/2} \hat{f}(\xi)]\|_{L^p(\mathbb{R}^n)}$, where $\wedge$ and $\mathscr{F}^{-1}$ denote the Fourier transform and its inverse. For $\| \cdot \|_{0,2}$ we simply write $\| \cdot \|$. Then in [2], Th. 7, it is shown that for $S \in \mathbb{N} \cup \{0\}$ the space $H^{S,p}(\mathbb{R}^n)$ coincides with the space of $L^p(\mathbb{R}^n)$-functions having distributional derivatives in $L^p(\mathbb{R}^n)$ up to order S. Moreover the following imbedding theorems of Sobolev type holds ([2], Th. 5 and 6):

(a) $H^{S+\epsilon,p} \subset H^{S,p}$ for $S \in \mathbb{R}^+$, $\epsilon > 0$, $1 < p < \infty$.

(b) $H^{S,p}(\mathbb{R}^n) \subset H^{t,q}(\mathbb{R}^n)$, $1 < p \leqslant q < \infty$, if $S > t$ and $1/q \geqslant 1/p - (S - t)/n$.

(c) $H^{S,p}(\mathbb{R}^n) \subset C^k(\mathbb{R}^n, \mathbb{R})$ for $S > 0$, $1 < p < \infty$, $k \in \mathbb{N} \cup \{0\}$, if $S > k + n/p$.

Finally, $H^{N,\infty}(\mathbb{R}^n)$ denotes for $N \in \mathbb{N} \cup \{0\}$ the space of $L^\infty(\mathbb{R}^n)$-functions with distributional derivatives up to order N in $L^\infty(\mathbb{R}^n)$ with norm $\| \cdot \|_{N,\infty}$.

1. A Nonlinearity with Second Spatial Derivatives

In this section we want to consider problems of the following form:

$$\begin{gathered}\frac{\partial^2 u}{\partial t^2} - \Delta \frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (\sigma_i(u_{x_i})) = 0 \\ u(x, 0) = \varphi(x), \qquad \frac{\partial u}{\partial t}(x, 0) = \psi(x).\end{gathered} \tag{1}$$

Here $\Delta := \sum_{i=1}^n (\partial^2/\partial x_i^2)$. The first lemma shows the local solvability of this initial value problem.

LEMMA 1.1. *Assume* $\sigma_i \in C^K_{\text{loc}}(\mathbb{R}, \mathbb{R})$, $\sigma_i(0) = 0$ $(i = 1,\dots n)$ *and* φ, $\psi \in H^{K,p}(\mathbb{R}^n)$, *where* $K \in \mathbb{N}$, $K > n/p + 1$, $K \geqslant 2$, $1 < p < \infty$. *Then problem* (1) *admits a unique solution* $u \in C^1([0, T_1], H^{K,p}(\mathbb{R}^n)) \cap C^2([0, T_1], H^{K-2,p}(\mathbb{R}^n))$, *where* $T_1 = 1/\omega(\|\varphi\|_{K,p}, \|\psi\|_{K,p})$ *with a function* $\omega \in C^0_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$.

Proof. Let A denote the operator $-\Delta + 1$ with domain $H^{2,p}(\mathbb{R}^n)$ and consider the integral equation

$$\begin{aligned} v(t) = e^{-tA}\psi &+ A^{-1} \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\sigma_i\left(\int_0^t v_{x_i}(\tau)\,d\tau + \varphi_{x_i}\right)\right) \\ &- A^{-1}e^{-tA} \sum_{i=1}^n \frac{\partial}{\partial x_i}(\sigma_i(\varphi_{x_i})) + \int_0^t e^{-(t-S)A} v(S)\,dS \qquad (1') \\ &- \int_0^t A^{-1}e^{-(t-S)A} \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\sigma_i'\left(\int_0^S v_{x_i}(\tau)\,d\tau + \varphi_{x_i}\right) v_{x_i}(S)\right) dS \equiv (\tilde{T}v)(t). \end{aligned}$$

It is easily shown that $\tilde{T}$ maps $B = C^0([0, T_1], H^{K,p}(\mathbb{R}^n))$ into itself for each $K > (n/p) + 1$. Moreover a straightforward computation involving essentially only the fact that a smooth function f maps each Sobolev space $H^{l,p}(\mathbb{R}^n)$ for $l > n/p$ locally lipschitzian into itself if $f(0) = 0$ shows that $\tilde{T}$ is a contraction mapping sending a suitable ball in B into itself. So Banach's fixed point theorem gives the existence of a unique solution v of the integral equation above. Defining $u(t) := \int_0^t v(S)\,dS + \varphi$ we arrive at the desired result.

Remarks. 1. This solution can be continued globally to each interval $[0, T]$, if an a-priori-estimate of $\| u_t(t)\|_{K,p}$ could be given.

2. If $K > n/p + 2$, then Sobolev's imbedding theorem shows that u is a classical solution of the Cauchy problem.

The desired a-priori-estimates are given in the following lemmas.

LEMMA 1.2. *Let u be the solution of* (1) *in the class* $C^1([0, T_1], H^{K,2}(\mathbb{R}^n)) \cap C^2([0, T_1], H^{K-2,2}(\mathbb{R}^n))$, $K > (n/2) + 2$. *Besides the assumptions of Lemma* 1.1 *let* $\sigma_i'(S) \geqslant 0$ *for all* $i = 1,\dots, n$ *and* $S \in \mathbb{R}$. *Then we have*

$$\| Au(t)\|^2 + \| u_t(t)\|^2 + \int_0^t \| A^{1/2}u_t(S)\|^2\,dS \leqslant \omega_1(t).\text{[1]}$$

Proof. Scalar multiplication of the differential equation by u_t gives

$$\frac{1}{2}\frac{d}{dt}\| u_t(t)\|^2 + \| A^{1/2}u_t(t)\|^2 - \int_{\mathbb{R}^n} \sum_{i=1}^n \frac{\partial}{\partial x_i}(\sigma_i(u_{x_i}))\, u_t\,dx = \| u_t(t)\|^2.$$

[1] $\omega_1, \omega_2, \dots$ denote functions in the class $C^0_{\text{loc}}(\mathbb{R}^+, \mathbb{R}^+)$, and $A := -\Delta + 1$ with $D(A) = H^{2,2}(\mathbb{R}^n)$.

Integrating by parts we have $-\int_{\mathbb{R}^n} (\partial/\partial x_i)(\sigma_i(u_{x_i}))\, u_t\, dx = (d/dt)\int_{\mathbb{R}^n} \sum_i (u_{x_i})\, dx$, where $\sum_i (S) := \int_0^S \sigma_i(\tau)\, d\tau \geqslant 0$ for all $S \in \mathbb{R}$. An integration over t shows

$$\| u_t(t)\|^2 + \int_0^t \| A^{1/2}u_t(S)\|^2\, dS \leqslant \omega_2(t).$$

Next we multiply the differential equation by Au and have

$$\frac{d}{dt}(u_t\,, Au) - \| A^{1/2}u_t\|^2 + \frac{1}{2}\frac{d}{dt}\| Au\|^2 - \sum_{i=1}^n \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i}(\sigma_i(u_{x_i}))\, Au\, dx$$
$$= (u_t\,, Au).$$

Integration by parts again gives

$$-\int_{\mathbb{R}^n} \frac{\partial}{\partial x_i}(\sigma_i(u_{x_i}))\, Au\, dx$$
$$= \sum_{j=1}^n \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i}(\sigma_i(u_{x_i})\, u_{x_jx_j})\, dx - \sum_{j=1}^n \sigma_i(u_{x_i})\, u_{x_ix_jx_j}\, dx - \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i}(\sigma_i(u_{x_i}))\, u\, dx$$
$$= -\sum_{j=1}^n \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j}(\sigma_i(u_{x_i})\, u_{x_ix_j})\, dx + \sum_{j=1}^n \int_{\mathbb{R}^n} \sigma_i'(u_{x_i})\, u_{x_ix_j}^2\, dx$$
$$-\int_{\mathbb{R}^n} \frac{\partial}{\partial x_i}(\sigma_i(u_{x_i})\, u)\, dx + \int_{\mathbb{R}^n} \sigma_i(u_{x_i})\, u_{x_i}\, dx$$
$$= \sum_{j=1}^n \int_{\mathbb{R}^n} \sigma_i'(u_{x_i})\, u_{x_ix_j}^2\, dx + \int_{\mathbb{R}^n} \sigma_i(u_{x_i})\, u_{x_i}\, dx \geqslant 0.$$

Therefore we have

$$\tfrac{1}{2}\| Au(t)\|^2 \leqslant \tfrac{1}{2}\| A\varphi\|^2 + |(u_t(t), Au(t))| + |(\psi, A\varphi)| + \int_0^t \| A^{1/2}u_t(S)\|^2\, dS$$
$$+ \tfrac{1}{2}\int_0^t (\| u_t(S)\|^2 + \| Au(S)\|^2)\, dS$$
$$\leqslant \tfrac{1}{2}\| A\varphi\|^2 + \| u_t(t)\|^2 + \tfrac{1}{4}\| Au(t)\|^2 + |(\psi, A\varphi)| + \int_0^t \| A^{1/2}u_t(S)\|^2\, dS$$
$$+ \tfrac{1}{2}\int_0^t (\| u_t(S)\|^2 + \| Au(S)\|^2)\, dS$$

and the result follows easily.

LEMMA 1.3. *Under the assumptions of the previous lemma we also have*

$$\int_0^t \| u_{tt}(S)\|^2\, dS + \| A^{1/2}u_t(t)\|^2 + \int_0^t \| Au_t(S)\|^2\, dS + \| A^{3/2}u(t)\|^2 \leqslant \omega_3(t),$$

if we additionally have $n = 2$ and $|\sigma_i''(S)| \leqslant C_1 |S|$ for $|S| \geqslant 1$, $C_2 |S|^2 \leqslant \sigma_i'(S)$ for $|S| \geqslant 1$.[2]

Proof. Multiplying the equation by A^2u we get

$$(A^{1/2}u_{tt}, A^{3/2}u) + \frac{1}{2}\frac{d}{dt}\| A^{3/2}u \|^2 - \sum_{i=1}^n \int_{\mathbb{R}^n} \sigma_i'(u_{x_i})\, u_{x_i x_i} A^2 u\, dx = (u_t, A^2u).$$

Now we have

$$A^2u = \sum_{j,k=1}^n u_{x_j x_j x_k x_k} - 2\sum_{i=1}^n u_{x_i x_i} + u$$

so that

$$\begin{aligned}
&- \int_{\mathbb{R}^n} \sigma_i'(u_{x_i})\, u_{x_i x_i} A^2 u\, dx \\
&= - \sum_{k,j=1}^n \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j}((\sigma_i(u_{x_i}))_{x_i}\, u_{x_j x_k x_k})\, dx + \sum_{k,j=1}^n \int_{\mathbb{R}^n} (\sigma_i(u_{x_i}))_{x_i x_j}\, u_{x_j x_k x_k}\, dx \\
&\quad + 2\sum_{j=1}^n \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j}((\sigma_i(u_{x_i}))_{x_i}\, u_{x_j})\, dx - 2\sum_{j=1}^n \int_{\mathbb{R}^n} (\sigma_i(u_{x_i}))_{x_i x_j}\, u_{x_j}\, dx \\
&\quad - \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i}(\sigma_i(u_{x_i})\, u)\, dx + \int_{\mathbb{R}^n} \sigma_i(u_{x_i})\, u_{x_i}\, dx \\
&= \sum_{k,j=1}^n \int_{\mathbb{R}^n} \frac{\partial}{\partial x_k}((\sigma_i(u_{x_i}))_{x_i x_j}\, u_{x_j x_k})\, dx - \sum_{j,k=1}^n \int_{\mathbb{R}^n} (\sigma_i(u_{x_i}))_{x_i x_j x_k}\, u_{x_j x_k}\, dx \\
&\quad - 2\sum_{j=1}^n \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i}((\sigma_i(u_{x_i}))_{x_j}\, u_{x_j})\, dx + 2\sum_{j=1}^n \int_{\mathbb{R}^n} \sigma_i'(u_{x_i})\, u_{x_j x_i}^2\, dx + \int_{\mathbb{R}^n} \sigma_i(u_{x_i})\, u_{x_i}\, dx \\
&= - \sum_{j,k=1}^n \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i}((\sigma_i(u_{x_i}))_{x_j x_k}\, u_{x_j x_k})\, dx + \sum_{j,k=1}^n \int_{\mathbb{R}^n} (\sigma_i(u_{x_i}))_{x_j x_k}\, u_{x_i x_j x_k}\, dx \\
&\quad + 2\sum_{j=1}^n \int_{\mathbb{R}^n} \sigma_i'(u_{x_i})\, u_{x_i x_j}^2\, dx + \int_{\mathbb{R}^n} \sigma_i(u_{x_i})\, u_{x_i}\, dx \\
&= \sum_{j,k=1}^n \int_{\mathbb{R}^n} (\sigma_i'(u_{x_i})\, u_{x_i x_j x_k}^2 + \sigma_i''(u_{x_i})\, u_{x_i x_k} u_{x_i x_j} u_{x_i x_j x_k})\, dx \\
&\quad + 2\sum_{j=1}^n \int_{\mathbb{R}^n} \sigma_i'(u_{x_i})\, u_{x_i x_j}^2\, dx + \int_{\mathbb{R}^n} \sigma_i(u_{x_i})\, u_{x_i}\, dx
\end{aligned}$$

[2] $C_1, C_2, \ldots$ denote positive constants.

Moreover it is $(A^{1/2}u_{tt}, A^{3/2}u) = (d/dt)(A^{1/2}u_t, A^{3/2}u) - \| Au_t \|^2$, so that the following identity holds:

$$\frac{d}{dt}(A^{1/2}u_t, A^{3/2}u) - \| Au_t \|^2 + \frac{1}{2}\frac{d}{dt}\| A^{3/2}u \|^2 + \sum_{i,j,k=1}^{n} \int_{\mathbb{R}^n} \sigma_i'(u_{x_i})\, u^2_{x_ix_jx_k}\, dx$$
$$+ \sum_{i,j,k=1}^{n} \int_{\mathbb{R}^n} \sigma_i''(u_{x_i})\, u_{x_ix_k}u_{x_ix_j}u_{x_ix_jx_k}\, dx + 2\sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \sigma_i'(u_{x_i})\, u^2_{x_ix_j}\, dx \tag{2}$$
$$+ \sum_{i=1}^{n} \int_{\mathbb{R}^n} \sigma_i(u_{x_i})\, u_{x_i}\, dx = (u_t, A^2u).$$

Multiplying the differential equation by $Au_t + u_{tt}$ and adding half of the equation (2) we have

$$\| u_{tt} \|^2 + \frac{d}{dt}\| A^{1/2}u_t \|^2 + \frac{1}{2}\| Au_t \|^2 + \frac{1}{4}\frac{d}{dt}\| A^{3/2}u \|^2$$
$$+ \tfrac{1}{2}\sum_{i,j,k=1}^{n} \int_{\mathbb{R}^n} \sigma_i'(u_{x_i})\, u^2_{x_ix_jx_k}\, dx$$
$$+ \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \sigma_i'(u_{x_i})\, u^2_{x_ix_j}\, dx + \tfrac{1}{2}\sum_{i=1}^{n} \int_{\mathbb{R}^n} \sigma_i(u_{x_i})\, u_{x_i}\, dx$$
$$= -\frac{1}{2}\frac{d}{dt}(A^{1/2}u_t, A^{3/2}u) - \frac{1}{2}\sum_{i,j,k=1}^{n} \int_{\mathbb{R}^n} \sigma_i''(u_{x_i})\, u_{x_ix_k}u_{x_ix_j}u_{x_ix_jx_k}\, dx$$
$$+ \sum_{i=1}^{n} \int_{\mathbb{R}^n} \sigma_i'(u_{x_i})\, u_{x_ix_i}Au_t\, dx + \sum_{i=1}^{n} \int_{\mathbb{R}^n} \sigma_i'(u_{x_i})\, u_{x_ix_i}u_{tt}\, dx$$
$$+ (u_t, \tfrac{1}{2}A^2u + Au_t + u_{tt}).$$

An integration over t gives the inequality

$$\int_0^t \| u_{tt}(S)\|^2\, dS + \| A^{1/2}u_t(t)\|^2 + \int_0^t \| Au_t(S)\|^2\, dS + \tfrac{1}{4}\| A^{3/2}u(t)\|^2$$
$$+ \tfrac{1}{2}\sum_{i,j,k=1}^{n} \int_0^t\int_{\mathbb{R}^n} \sigma_i'(u_{x_i})\, u^2_{x_ix_jx_k}\, dx\, dS$$
$$\leq \| A^{1/2}\psi \|^2 + \tfrac{1}{4}\| A^{3/2}\varphi \|^2 + \tfrac{1}{2}\int_{\mathbb{R}^n} | A^{1/2}u_t(t)\, A^{3/2}u(t)|\, dx$$
$$+ \tfrac{1}{2}\int_{\mathbb{R}^n} | A^{1/2}\psi A^{3/2}\varphi |\, dx + \tfrac{1}{2}\sum_{i,j,k=1}^{n} \int_0^t\int_{\mathbb{R}^n} | \sigma_i''(u_{x_i})\, u_{x_ix_k}u_{x_ix_j}u_{x_ix_jx_k} |\, dx\, dS$$

$$+ \sum_{i=1}^{n} \int_0^t \int_{\mathbb{R}^n} | \sigma_i'(u_{x_i}) u_{x_i x_i} A u_t | \, dx \, dS$$

$$+ \sum_{i=1}^{n} \int_0^t \int_{\mathbb{R}^n} | \sigma_i'(u_{x_i}) u_{x_i x_i} u_{tt} | \, dx \, dS + \int_0^t |(u_t \, , \tfrac{1}{2} A^2 u + A u_t + u_{tt})| \, dS. \tag{3}$$

The terms on the right side are estimated in the following way:

$$\tfrac{1}{2} \int_{\mathbb{R}^n} | A^{1/2} u_t(t) \, A^{3/2} u(t)| \, dx \leqslant \tfrac{3}{8} \| A^{1/2} u_t(t) \|^2 + \tfrac{1}{6} \| A^{3/2} u(t) \|^2,$$

$$\begin{aligned} \int_{\mathbb{R}^n} | \sigma_i'(u_{x_i}) u_{x_i x_i} u_{tt} | \, dx &\leqslant C_3 \int_{\mathbb{R}^n} (1 + | u_{x_i} |^2) \, | u_{x_i x_i} u_{tt} | \, dx \\ &\leqslant C_4 (\| u_{x_i} \|_{0,8}^2 \| u_{x_i x_i} \|_{0,4} \| u_{tt} \| + \| u_{x_i x_i} \| \| u_{tt} \|) \\ &\leqslant C_5 (\| A u \|^2 \| A^{3/2} u \| \| u_{tt} \| + \| A u \| \| u_{tt} \|) \end{aligned}$$

by Sobolev's theorem.

Lemma 2 then gives

$$\frac{1}{2} \int_{\mathbb{R}^n} | A^{1/2} u_t(t) \, A^{3/2} u(t)| \, dx \leqslant \frac{1}{2n} \| u_{tt} \|^2 + \omega_4(t) \| A^{3/2} u \|^2.$$

In the same way we get

$$\int_{\mathbb{R}^n} | \sigma_i'(u_{x_i}) u_{x_i x_i} A u_t | \, dx \leqslant \frac{1}{2n} \| A u_t \|^2 + \omega_4(t) \| A^{3/2} u \|^2.$$

Furthermore we have by use of the calculus inequalities (see e.g. [4], p. 24):

$$\begin{aligned} &\int_{\mathbb{R}^n} | \sigma_i''(u_{x_i}) u_{x_i x_k} u_{x_i x_j} u_{x_i x_j x_k} | \, dx \\ &\leqslant C_6 \left(\int_{|u_{x_i}|<1} | u_{x_i x_k} u_{x_i x_j} u_{x_i x_j x_k} | \, dx + \int_{|u_{x_i}|\geqslant 1} | u_{x_i} u_{x_i x_k} u_{x_i x_j} u_{x_i x_j x_k} | \, dx \right) \\ &\leqslant C_6 \| u_{x_i x_k} \|_{0,4} \| u_{x_i x_j} \|_{0,4} \| u_{x_i x_j x_k} \| + \frac{C_2}{4} \int_{|u_{x_i}|\geqslant 1} u_{x_i}^2 u_{x_i x_j x_k}^2 \, dx \\ &\quad + C_7 \int_{\mathbb{R}^n} u_{x_i x_k}^2 u_{x_i x_j}^2 \, dx \\ &\leqslant C_8 \| A u \| \| A^{3/2} u \|^2 + \tfrac{1}{4} \int_{\mathbb{R}^n} \sigma_i'(u_{x_i}) \, u_{x_i x_j x_k}^2 \, dx. \end{aligned}$$

Going back to (3) we arrive at

$$\tfrac{1}{2}\int_0^t \| u_{tt}(S)\|^2\, dS + \tfrac{1}{8}\| A^{1/2}u_t(t)\|^2 + \tfrac{1}{2}\int_0^t \| Au_t(S)\|^2\, dS + \tfrac{1}{12}\| A^{3/2}u(t)\|^2$$

$$\leqslant \| A^{1/2}\psi \|^2 + \tfrac{1}{4}\| A^{3/2}\varphi \|^2 + \tfrac{1}{2}\| A^{1/2}\psi \|\, \| A^{3/2}\varphi \| + \int_0^t \omega_5(S)\, \| A^{3/2}u(S)\|^2\, dS$$

$$+ C_8' \int_0^t \| A^{1/2}u_t(S)\|^2\, dS + \tfrac{1}{4}\int_0^t \| u_{tt}(S)\|^2\, dS.$$

Gronwall's lemma gives the claimed result.

Remark. The last assumption on σ_i can be replaced by $|\sigma_i''(S)| \leqslant C_9$ for all $S \in \mathbb{R}$, as can be seen from the proof.

THEOREM 1.1. *Assume* $K \in \mathbb{N}$, $K > (n/2) + 1$, $n = 1$ *or* $n = 2$, $\sigma_i \in C^K_{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$, $\sigma_i(0) = 0$ $(i = 1,..., n)$, *and* $\varphi, \psi \in H^{K,2}(\mathbb{R}^n)$. *If* $\sigma_i'(S) \geqslant 0$ *for* $i = 1,..., n$; $S \in \mathbb{R}$, *and moreover in the case* $n = 2$: *either* $|\sigma_i''(S)| \leqslant C_1 |S|$ *for* $|S| \geqslant 1$ *and* $C_2 |S|^2 \leqslant \sigma_i'(S)$ *for* $|S| \geqslant 1$ *or* $|\sigma_i''(S)| \leqslant$ const., *then problem* (1) *admits a unique solution* u *in the class* $C^1([0, T], H^{K,2}(\mathbb{R}^n)) \cap C^2([0, T], H^{K-2,2}(\mathbb{R}^n))$ *for any* $T > 0$.

Proof. The integral equation (1′) gives the following inequality:

$$\| u_t(t)\|_{2,2} \leqslant C_{10}\Big(\| \psi \|_{2,2} + \sum_{i=1}^n \| \sigma_i'(u_{x_i}(t))\, u_{x_ix_i}(t)\| + \sum_{i=1}^n \| \sigma_i'(\varphi_{x_i})\, \varphi_{x_ix_i} \|$$

$$+ \int_0^t \| u_t(S)\|_{2,2}\, dS + \int_0^t \frac{1}{(t-S)^{1/2}} \| \sigma_i'(u_{x_i})\, u_{x_it} \|\, dS\Big).$$

An application of Sobolev's imbedding theorem leads to

$$\| \sigma_i'(u_{x_i})\, u_{x_ix_i} \| \leqslant \| \sigma_i'(u_{x_i})\|_{0,\infty}\, \| u_{x_ix_i} \| \leqslant \omega_6(\| A^{n/2}u \|)\, \| u \|_{2,2},$$

and

$$\| \sigma_i'(u_{x_i})\, u_{x_it} \| \leqslant \omega_6(\| A^{n/2}u \|)\, \| u_t \|_{1,2},$$

so that Lemma 2 and 3 and Growall's lemma give an a-priori-estimate of $\| u_t(t)\|_{2,2}$. In the same way it is possible to show step by step any desired a-priori-bound which gives together with Lemma 1.1 the claimed result.

EXAMPLE. $\partial^2u/\partial t^2 - \Delta(\partial u/\partial t) - \sum_{i=1}^2 u_{x_i}^2 u_{x_ix_i} = 0$ for $n = 2$. Cf. also the papers of Greenberg–MacCamy–Mizel [6], Greenberg [5], and Cleménts [3].

Remark. Theorem 1.1 also holds if there is an additional term $f(u)$ on the left of the differential equation where f fulfills: $f \in C^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$, $f(0) = 0$, and

$F(S) := \int_0^S f(\sigma)\, d\sigma \geqslant -\text{const}.\ S^2$ for all $S \in \mathbb{R}$, and moreover for $n = 2$ either f has polynomial growth or f is monotonically increasing. The method of proof is the same.

2. Nonlinearities Involving Spatial and Time Derivatives

In this paragraph initial value problems of the following type are considered:

$$\begin{gathered}\frac{\partial^2 u}{\partial t^2} + A_p \frac{\partial u}{\partial t} + F(u, u_{x_1}, \ldots, u_{x_n}, u_t, u_{tx_1}, \ldots, u_{tx_n}) = 0 \\ u(x, 0) = \varphi(x), \qquad \frac{\partial u}{\partial t}(x, 0) = \psi(x).\end{gathered} \tag{4}$$

Here we define

$$A_p u = - \sum_{|\alpha|, |\beta| \leqslant 1} D^\alpha(A_\alpha{}^\beta(x)\, D^\beta u) \qquad \text{for } u \in D(A_p) = H^{2,p}(\mathbb{R}^n),$$

where $A_{\alpha\beta} \in C^\infty(\mathbb{R}^n)$ and $\sum_{|\alpha|=|\beta|=1} A_{\alpha\beta}(x)\, \zeta^{\alpha+\beta} \geqslant C_1 \mid \zeta \mid^2$ for $\zeta \in \mathbb{R}^n$, $x \in \mathbb{R}^n$. For our purposes we may assume without loss of generality that $(A_2 u, u) \geqslant C_2 \| u \|_{1,2}^2$ holds with a positive constant C_2. F is a mapping from $\mathbb{R}^{2n+2}$ into $\mathbb{R}$ having the following properties:

$$\begin{aligned} &F(a, b_1, \ldots, b_n, C, d_1, \ldots, d_n) \\ &\quad = f(a, b_1, \ldots, b_n, C, d_1, \ldots, d_n) + f_2(a) \\ &\qquad + \sum_{i=1}^n (g_i'(a)\, b_i C + 2 g_i(a)\, d_i) + h(a, b_1, \ldots, b_n, C, d_1, \ldots, d_n). \end{aligned}$$

Here $f_1, f_2, g_i, h \in C_{\mathrm{loc}}^\infty$, $f_1(0) = f_2(0) = h(0) = 0$. Moreover we assume $f_1(a, b_1, \ldots, b_n, C, d_1, \ldots, d_n) \cdot C \geqslant -\text{const}(a^2 + \sum_{i=1}^n b_i^2 + C^2 + \sum_{i=1}^n \mid d_i \mid^{2-\epsilon} \times (\mid a \mid^\epsilon + \sum_{i=1}^n \mid b_i \mid^\epsilon + \mid C \mid^\epsilon))$ for all $(a, \ldots, d_n) \in \mathbb{R}^{2n+2}$ with a positive ϵ, $\tilde{F}(S) := \int_0^S f_2(\sigma)\, d\sigma \geqslant -\text{const}\ S^2$ for all $S \in \mathbb{R}$

$$\mid h(a, b_1, \ldots, b_n, C, d_1, \ldots, d_n) \mid \leqslant \text{const} \left(\mid a \mid + \sum_{i=1}^n \mid b_i \mid + \mid C \mid + \sum_{i=1}^n \mid d_i \mid \right).$$

Finally $\varphi, \psi \in H^{K,2}(\mathbb{R}^n)$.

In the following we shall always assume that these conditions are fulfilled. Considering the associated equation

$$u_t(t) = e^{-tA_p}\psi - \int_0^t e^{-(t-S)A_p} F(u(S), \ldots, u_{tx_n}(S))\, dS \tag{5}$$

it is not hard to show by means of Banach's fixed point theorem the existence of a unique local solution u of (4) in the class $C^1([0, T_1], H^{K,2}(\mathbb{R}^n)) \cap C^2([0, T_1], H^{K-2,2}(\mathbb{R}^n))$, if $K > (n/2) + 1$, $K \geqslant 2$. As in the preceding paragraph one has only to use the fact that $H^{K,2}(\mathbb{R}^n)$ is a Banach algebra for $K > n/2$. From now on we always assume $K \geqslant n/2 + [n/2] + 3$, so that the formal calculations are correct. In order to show the existence of a global solution with the same regularity we have to give an a-priori-estimate of $\| u_t(t)\|_{K,2}$, which is done step-by-step in the following lemmas.

LEMMA 2.1. *The following a-priori-estimate holds*

$$\| u_t(t)\|^2 + \int_0^t \| u_t(S)\|_{1,2}^2 \, dS \leqslant \omega_1(t).$$

Proof. A scalar multiplication of the differential equation with u_t gives

$$\frac{1}{2}\frac{d}{dt}\| u_t(t)\|^2 + \| A^{1/2}u_t(t)\|^2 + \int_{\mathbb{R}^n} f_1(u, \nabla u, u_t, \nabla u_t)\, dx + \frac{d}{dt}\int_{\mathbb{R}^n} \tilde{F}(u(t))\, dx$$

$$\leqslant C_2 \int_{\mathbb{R}^n} \left(| u(t)| + \sum_{i=1}^n | u_{x_i}(t)| + | u_t(t)| + \sum_{i=1}^n | u_{x_it}(t)|\right) | u_t(t)| \, dx.$$

because we have

$$\int_{\mathbb{R}^n} (g_i'(u)\, u_{x_i}u_t + 2g_i(u)\, u_{x_it})\, u_t \, dx = \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i}(g_i(u)\, u_t^2)\, dx = 0.$$

Using the assumptions on f_1 and $\tilde{F}$ we arrive after integration at

$$\begin{aligned} &\tfrac{1}{2}\| u_t(t)\|^2 + \int_0^t \| A^{1/2}u_t(S)\|^2 \, dS \\ &\quad \leqslant \tfrac{1}{2}\| \psi \|^2 + \int_{\mathbb{R}^n} \tilde{F}(\varphi)\, dx + \tfrac{1}{2}\int_0^t \| A^{1/2}u_t(S)\|^2 \, dS \\ &\qquad + C_3\left(\| u(t)\|^2 + \int_0^t \left(\| u(S)\|^2 + \sum_{i=1}^n \| u_{x_i}(S)\|^2 + \| u_t(S)\|^2\right) dS\right). \end{aligned}$$

The identities $u(t) = \varphi + \int_0^t u_t(S)\, dS$ and $u_{x_i}(t) = \varphi_{x_i} + \int_0^t u_{x_it}(S)\, dS$ imply:

$$\| u(t)\|^2 \leqslant 2\left(\| \varphi \|^2 + t\int_0^t \| u_t(S)\|^2 \, dS\right),$$

and

$$\int_0^t \| u(S)\|^2 \, dS \leqslant 2t \| \varphi \|^2 + C_3' t \int_0^t \left(\int_0^S \| A^{1/2}u_t(\sigma)\|^2 \, d\sigma\right) dS,$$

as well as

$$\int_0^t \| u_{x_i}(S)\|^2\, dS \leqslant 2t \,\| \varphi_{x_i} \|^2 + C_3'' t \int_0^t \left(\int_0^S \| A^{1/2} u_t(\sigma)\|^2\, d\sigma \right) dS.$$

Therefore we finally have a linear integral inequality of Volterra type, so that a generalization of Gronwall's lemma gives the desired result.

LEMMA 2.2 (Calculus inequality). *The following inequality holds* $\| \nabla u \|_{0,p} \leqslant C_5 \| u \|_{K,q}^{a} \| u \|_{0,2}^{1-a}$ *provided we have* $n \geqslant 3$, $K = 2 - \epsilon$ ($\epsilon > 0$ *small enough*), $q = \min(n/2, 2)$, $p > 2$, $1/K < a < 1$, $a > \frac{2}{3}$ *in the case* $n = 3$, *and*

$$\frac{1}{p} = \frac{1}{n} + a\left(\frac{1}{q} - \frac{K}{n}\right) + (1-a)\frac{1}{2}. \tag{6}$$

Proof. If $\lambda := 2q/(qn - 2n + 2qk)$, then $0 > 1/\lambda = n/2 - n/q + K > 1$. Setting $\kappa := (a - \lambda)/(1 - \lambda)$, we easily check from (6): $-n/p + 1 + n/2 = a/\lambda > 1$, using $p > 2$, so that $0 < \kappa < 1$. Hölder's inequality leads to $\| \nabla u \|_{0,p} \leqslant \| \nabla u \|_{0,p\tilde{p}\kappa}^{\kappa} \| \nabla u \|_{0,p\tilde{q}(1-\kappa)}^{1-\kappa}$, where $1/\tilde{p} + 1/\tilde{q} = 1$. We set $\tilde{q} := 2/p(1-\kappa)$ and want to have $\tilde{q} > 1$. To this end we calculate as follows:

(a) In the case q $= 2$ we get from (6): $p = 2n/(2 + n - 2aK)$. From $1 - \kappa = K(1 - a)/(K - 1)$ we conclude $p(1 - \kappa) < 2$ iff $nK(1 - a) < (2 + n - 2aK)(K - 1)$. This is equivalent to $0 < (n + 2(1 - K))(Ka - 1)$ which is obviously fulfilled.

(b) In the case $q = n/2$, i.e. $n = 3$, we directly calculate

$$p(1-\kappa) = 1\Big/\left(\frac{5}{6} - \frac{1}{2}a + \frac{\epsilon}{3}a\right) \cdot (1-a)\Big/\left(1 - \frac{2}{3 - 2\epsilon}\right) < 2,$$

because $a > \frac{2}{3}$. In any case we therefore have $\tilde{q} > 1$ and $\tilde{p} = 2/[2 - p(1-\kappa)]$. So we arrive at $\| \nabla u \|_{0,p} \leqslant \| \nabla u \|_{0,2p\kappa/[2-p(1-\kappa)]}^{\kappa} \| \nabla u \|_{0,2}^{1-\kappa}$. Using Fourier transforms the last term can be treated in the following way:

$$\| \nabla u \|_{0,2} \leqslant \left(\int_{\mathbb{R}^n} (|\xi|^2 |\hat{u}(\xi)|^{2\lambda})^{\hat{p}}\, d\xi \right)^{1/2\hat{p}} \left(\int_{\mathbb{R}^n} |\hat{u}(\xi)|^{2\hat{q}(1-\lambda)}\, d\xi \right)^{1/2\hat{q}}.$$

With $\hat{q} := 1/(1-\lambda)$ and $\hat{p} := 1/\lambda$ this means: $\| \nabla u \|_{0,2} \leqslant C_6 \| u \|_{1/\lambda,2}^{\lambda} \| u \|_{0,2}^{1-\lambda}$. Summarizing we have $\| \nabla u \|_{0,p} \leqslant C_6 \| \nabla u \|_{0,2p\kappa/[2-p(1-\kappa)]}^{\kappa} \| u \|_{1/\lambda,2}^{\lambda(1-\kappa)} \| u \|_{0,2}^{(1-\kappa)(1-\lambda)}$. A direct calculation gives $\kappa(1/q - \frac{1}{2} - (K-1)/n) + \frac{1}{2} = 1/p$, so that $[2 - p(1-\kappa)]/2p\kappa = 1/q - (K-1)/n$, which means by Sobolev's imbedding theorem $H^{K-1,q} \subset L^{2p\kappa/[2-p(1-\kappa)]}$. Moreover it is $\frac{1}{2} = 1/p - (K - 1/\lambda)/n$, therefore $H^{K,q} \subset H^{1/\lambda,2}$. Finally we have $\kappa + (1-\kappa)\lambda = a$ and $(1-\kappa)(1-\lambda) = 1 - a$, which completes the proof.

LEMMA 2.3. *Assume* $f(u, u_{x_1}, \ldots, u_{x_n}, u_t, u_{tx_1}, \ldots, u_{tx_n}) = \sum_{j=1}^{N} f_1^{(j)}(u, u_{x_1}, \ldots, u_{x_n}) f_2^{(j)}(u_t, u_{tx_1}, \ldots, u_{tx_n})$ *satisfies the following growth conditions:*

$$|f_1^{(j)}(u, u_{x_1}, \ldots, u_{x_n})| = C_7 \left(|u|^{\sigma_0^{(j)}} + \sum_{i=1}^{n} |u_{x_i}|^{\sigma_1^{(j)}} \right)$$

$$|f_2^{(j)}(u_t, u_{tx_1}, \ldots, u_{tx_n})| \leqslant C_8 \left(|u_t|^{\sigma_2^{(j)}} + \sum_{i=1}^{n} |u_{tx_i}|^{\sigma_3^{(j)}} \right)$$

with $\sigma_0^{(j)} < \infty$ *in the case* $n = 2$,

$$\sigma_0^{(j)} \left(\frac{1}{2} - \frac{1}{n} \right) + \sigma_2^{(j)} \frac{1}{2} < \frac{2}{n} + \frac{1}{2}, \qquad \textit{when} \qquad \sigma_2^{(j)} \geqslant 1,$$

$$\sigma_0^{(j)} + \sigma_2^{(j)} < \frac{n+2}{n-2}, \qquad \textit{when} \qquad \sigma_2^{(j)} < 1,$$

$$\sigma_0^{(j)} \left(\frac{1}{2} - \frac{1}{n} \right) + \sigma_3^{(j)} \left(\frac{1}{2} + \frac{1}{n} \right) < \frac{2}{n} + \frac{1}{2}, \qquad \textit{when} \qquad \sigma_3^{(j)} \geqslant 1,$$

$$\sigma_0^{(j)} \left(\frac{1}{2} - \frac{1}{n} \right) + \frac{\sigma_3^{(j)}}{2} < \frac{1}{n} + \frac{1}{2}, \qquad \textit{when} \qquad \sigma_3^{(j)} < 1,$$

$$\sigma_1^{(j)} + \sigma_2^{(j)} < \frac{n+4}{n}, \qquad \textit{when} \qquad \sigma_2^{(j)} \geqslant 1,$$

$$\sigma_1^{(j)} + \left(1 - \frac{2}{n} \right) \sigma_2^{(j)} < \frac{n+2}{n}, \qquad \textit{when} \qquad \sigma_2^{(j)} < 1,$$

$$\frac{\sigma_1^{(j)}}{2} + \sigma_3^{(j)} \left(\frac{1}{2} + \frac{1}{n} \right) < \frac{2}{n} + \frac{1}{2}, \qquad \textit{when} \qquad \sigma_3^{(j)} \geqslant 1,$$

$$\sigma_1^{(j)} + \sigma_3^{(j)} < \frac{n+2}{n}, \qquad \textit{when} \qquad \sigma_3^{(j)} < 1,$$

$$\sigma_0^{(j)} + \sigma_2^{(j)} \geqslant 1, \qquad \sigma_0^{(j)} + \sigma_3^{(j)} \geqslant 1, \qquad \sigma_1^{(j)} + \sigma_2^{(j)} \geqslant 1,$$
$$\sigma_1^{(j)} + \sigma_3^{(j)} \geqslant 1.$$

Moreover let $g_i(u)$ *vanish identically for* $n \geqslant 4$, *otherwise assume* $|g_i'(u)| \leqslant C_9(1 + |u|^{\rho_0^{(i)}})$, *with* $\rho_0^{(i)} < \infty$ *for* $n = 2$, *and* $\rho_0^{(i)} < 1$ *for* $n = 3$. *Then the following a-priori-estimate holds:* $\| u_t(t) \|_{2-\epsilon, p} \leqslant \omega_2(t)$ *for* $p = n/2$ *and any* $\epsilon > 0$.

Proof. Using (6) and the well-known estimates for the operator e^{-tA_p} (see e.g. [4], p. 160) we arrive at the inequality (for any $1 < p < \infty$):

$$\| u_t(t) \|_{2-\epsilon, p} \leqslant C_{10} \left(\| \psi \|_{2,p} + \int_0^t \frac{1}{(t-S)^{1-\epsilon/2}} \| F(u(S), \nabla u(S), u_t(S), \nabla u_t(S)) \|_{0,p} \, dS \right) \tag{7}$$

We have to estimate the integrand and treat the terms separately (suppressing the indices j).

We assume $n > 2$, because $n \leqslant 2$ can be handled similarly.

(a) The term $\| \, |u|^{\sigma_0} \, |u_t|^{\sigma_2} \|_{0,p}$ for $p = n/2$ or $p = \min(2, n/2)$ is handled as follows.

(α) In the case $\sigma_2 \geqslant 1$ we have

$$\| \, |u|^{\sigma_0} \, |u_t|^{\sigma_2} \|_{0,p} \leqslant \| u \|^{\sigma_0}_{0,p\tilde{p}\sigma_0} \| u_t \|^{\sigma_2 - 1}_{0,p\tilde{q}(\sigma_2-1)} \| u_t \|_{0,p\tilde{r}}$$

where

$$0 \leqslant \frac{1}{\tilde{q}} := \frac{p}{2}(\sigma_2 - 1) < \frac{2p}{n} \leqslant 1, \qquad \text{i.e.} \qquad p\tilde{q}(\sigma_2 - 1) = 2.$$

$$0 - \frac{1}{\tilde{p}} := p\sigma_0 \left(\frac{1}{2} - \frac{1}{n} \right) \leqslant \frac{2p}{n} \leqslant 1, \qquad \text{i.e.} \qquad \frac{1}{p\tilde{p}\sigma_0} = \frac{1}{2} - \frac{1}{n},$$

and

$$1 \geqslant \frac{1}{\tilde{r}} := 1 - \frac{1}{\tilde{p}} - \frac{1}{\tilde{q}} > 1 - \frac{2p}{n} \geqslant 0,$$

so that

$$\frac{1}{p\tilde{r}} \geqslant \frac{1}{p} - \frac{2 - \epsilon}{n}$$

using the growth restrictions, Sobolev's imbedding theorem then leads to

$$\| \, |u|^{\sigma_0} \, |u_t|^{\sigma_2} \|_{0,p} \leqslant C_{11} \| u \|^{\sigma_0}_{1,2} \| u_t \|^{\sigma_2 - 1}_{0,2} \| u_t \|_{2-\epsilon,p} .$$

(β) In the case $\sigma_2 < 1$ we have

$$\| \, |u|^{\sigma_0} \, |u_t|^{\sigma_2} \|_{0,p} \leqslant \| u \|^{\sigma_0 + \sigma_2 - 1}_{0,p\tilde{p}(\sigma_0 + \sigma_2 - 1)} \| u_t \|^{\sigma_2}_{0,p\tilde{q}\sigma_2} \| u \|^{1-\sigma_2}_{0,p\tilde{r}(1-\sigma_2)} .$$

Setting $1/\tilde{p} := p(\sigma_0 + \sigma_2 - 1)(\frac{1}{2} - 1/n) < 1$, i.e. $1/[p\tilde{p}(\sigma_0 + \sigma_2 - 1)] = \frac{1}{2} - 1/n$, and furthermore $\tilde{q}$, $\tilde{r}$ in such a way that $1 \geqslant 1/\tilde{q} + 1/\tilde{r} = 1 - 1/\tilde{p} \geqslant \epsilon$, and $\sigma_2 \geqslant 1/\tilde{q} \geqslant p\sigma_2(1/p - (2 - \epsilon)/n)$, $1 - \sigma_2 \geqslant 1/\tilde{r} \geqslant p(1 - \sigma_2)(1/p - (2 - \epsilon)/n)$. This choice is possible because $1/\tilde{p} \leqslant p(2 - 2\epsilon)/n < 1 - p(1/p - (2 - \epsilon)/n)$, using the growth condition. The imbedding theorem then gives the estimate

$$\| \, |u|^{\sigma_0} \, |u_t|^{\sigma_2} \|_{0,p} \leqslant C_{12} \| u \|^{\sigma_0 + \sigma_2 - 1}_{1,2} \| u_t \|^{\sigma_2}_{2-\epsilon,p} \| u \|^{1-\sigma_2}_{2-\epsilon,p} .$$

(b) The second term which has to be treated is $\| \, |u|^{\sigma_0} \, |u_{tx_i}|^{\sigma_3} \|_{0,p}$.

(α) If $\sigma_3 \geqslant 1$, we conclude as follows:

$$\| \, |u|^{\sigma_0} \, |u_{tx_i}|^{\sigma_3} \|_{0,p} \leqslant C_{13}(\| \, |u|^{\sigma_0} \, |u_{tx_i}| \, \|_{0,p} + \| \, |u|^{\sigma_0} \, |u_{tx_i}|^{\bar{\sigma}_3} \|_{0,p}), \tag{8}$$

where $\sigma_0(\frac{1}{2} - 1/n) + \bar{\sigma}_3(\frac{1}{2} + 1/n) = \frac{1}{2} + 2/n - \delta$, $\delta > 0$ sufficiently small. The first term of (8) is estimated by Hölder's inequality with exponents $\tilde{p}$ and $\tilde{q}$ for $p = n/2$ or $p = \min(2, n/2)$, setting $1/\tilde{p} := p\sigma_0(\frac{1}{2} - 1/n) < n/2(\frac{1}{2} + 2/n - \frac{1}{2} - 1/n) = \frac{1}{2}$ by using the growth condition, and $1/\tilde{q} := 1 - 1/\tilde{p}$ so that $1/p\tilde{q} = 1/p - \sigma_0(\frac{1}{2} - 1/n) \geqslant 1/p - (1 - \epsilon)/n$ for $\epsilon > 0$ sufficiently small. So we arrive by the imbedding theorem at $\| \, |u|^{\sigma_0} u_{tx_i} \|_{0,p} \leqslant C_{14} \| u \|_{1,2}^{\sigma_0} \| u_t \|_{2-\epsilon,p}$. The second term of (8) is treated for $p = \min(2, n/2)$ only by Hölder's inequality with exponents $\tilde{p}$, $\tilde{q}$ where $1/\tilde{p} := p\sigma_0(\frac{1}{2} - 1/n) < \frac{1}{2}$. The second factor $\| u_{tx_i} \|_{0,p\tilde{q}\bar{\sigma}_3}^{\bar{\sigma}_3}$ is estimated using Lemma 2.2. An easy calculation shows $p\tilde{q}\bar{\sigma}_3 > 2$, and setting $a = (1/\bar{\sigma}_3)(1/p - \frac{1}{2} - 2/n - \delta)/(1/p - \frac{1}{2} - 2/n + \epsilon/n) < 1/\bar{\sigma}_3$ we find $1/p\tilde{q}\sigma_3 = 1/n + a(1/p - (2 - \epsilon)/n) + (1 - a)\frac{1}{2}$ and $a > \frac{2}{3}$, so that Lemma 2.2 gives the inequality

$$\| u_{tx_i} \|_{0,p\tilde{q}\bar{\sigma}_3}^{\bar{\sigma}_3} \leqslant C_{15} \| u_t \|_{2-\epsilon,p}^{a\bar{\sigma}_3} \| u_t \|_{0,2}^{\bar{\sigma}_3(1-a)} \leqslant C_{16} \| u_t \|_{0,2}^{\bar{\sigma}_3(1-a)} (1 + \| u_t \|_{2-\epsilon,p})$$

Putting these results together we have

$$\| \, |u|^{\sigma_0} \, |u_{tx_i}|^{\sigma_3} \|_{0,p} \leqslant C_{17} \| u \|_{1,2}^{\sigma_0} (\| u_t \|_{2-\epsilon,p} + 1)\,(\|u_t\|_{0,2}^{\bar{\sigma}_3(1-a)} + 1)$$

for $p = \min(2, n/2)$.

(β) In the case $\sigma_3 < 1$ let p be either $n/2$ or $\min(2, n/2)$ and estimate as follows:

$$\| \, |u|^{\sigma_0} \, |u_{tx_i}|^{\sigma_3} \|_{0,p} \leqslant \| u \|_{0,p\tilde{p}(\sigma_0+\sigma_3-1)}^{\sigma_0+\sigma_3-1} \| u_{tx_i} \|_{0,p\tilde{q}\sigma_3}^{\sigma_3} \| u \|_{0,p\tilde{r}(1-\sigma_3)}^{1-\sigma_3} .$$

We define $1/\tilde{p} := p(\sigma_0 + \sigma_3 - 1)(\frac{1}{2} - 1/n) < 1$, and $\tilde{q}$, $\tilde{r}$ in such a way that $1 = 1/\tilde{p} + 1/\tilde{q} + 1/\tilde{r}$ and $1/p \geqslant 1/p\tilde{q}\sigma_3 \geqslant 1/p - (1 - \epsilon)/n$, $1/p \geqslant 1/p\tilde{r}(1 - \sigma_3) \geqslant 1/p - (2 - \epsilon)/n$. This is possible, because $1/\tilde{p} + \sigma_3(1 - (p/n)(1 - \epsilon)) + (1 - \sigma_3)(1 - (p/n)(2 - \epsilon)) < 1$ by use of the growth conditions. Therefore we are led to the inequality

$$\| \, |u|^{\sigma_0} \, |u_{tx_i}|^{\sigma_3} \|_{0,p} \leqslant C_{18} \| u \|_{1,2}^{\sigma_0+\sigma_3-1} \| u_t \|_{2-\epsilon,p}^{\sigma_3} \| u \|_{2-\epsilon,p}^{1-\sigma_3} .$$

(c) The next term *is* $\| \, |u_{x_i}|^{\sigma_1} \, |u_t|^{\sigma_2} \|_{0,p}$ for $p = n/2$ or $p = \min(2, n/2)$.

(α) If $\sigma_2 \geqslant 1$, Hölder's inequality gives if applied to the factors $|u_{x_i}|^{\sigma_1}$, $|u_t|^{\sigma_2 - 1}$ and u_t with exponents $\tilde{p}$, $\tilde{q}$, $\tilde{r}$ defined by $1/\tilde{p} := p\sigma_1/2 < 1$, $1/\tilde{q} := p(\sigma_2 - 1)/2 < 1$, and $1/\tilde{r} := 1 - 1/\tilde{p} - 1/\tilde{q} = 1 - (p/2)(\sigma_1 + \sigma_2 - 1) > 0$:

$$\| \, |u_{x_i}|^{\sigma_1} \, |u_t|^{\sigma_2} \|_{0,p} \leqslant C_{19} \| u \|_{1,2}^{\sigma_1} \| u_t \|_{0,2}^{\sigma_2 - 1} \| u_t \|_{2-\epsilon,p} ,$$

because the growth condition implies $1/p\tilde{r} > 1/p - (2 - \epsilon)/n$.

(β) If $\sigma_2 < 1$ we apply Hölder's inequality to the factors $| u_{x_i} |^{\sigma_1+\sigma_2-1}$, $| u_{x_i} |^{1-\sigma_2}$, and $| u_t |^{\sigma_2}$. The corresponding exponents $\tilde{p}$, $\tilde{q}$ and $\tilde{r}$ are restricted by:

$$\frac{1}{\tilde{p}} := \frac{p(\sigma_1 + \sigma_2 - 1)}{2} < 1, \qquad \frac{1}{p} \geqslant \frac{1}{p\tilde{q}(1-\sigma_2)} \geqslant \frac{1}{p} - \frac{1-\epsilon}{n},$$
$$\frac{1}{p} \geqslant \frac{1}{p\tilde{r}\sigma_2} \geqslant \frac{1}{p} - \frac{2-\epsilon}{n}.$$

This choice is possible for we easily calculate by the growth restriction: $1/\tilde{p} + (1-\sigma_2)(1-(p/n)(1-\epsilon)) + \sigma_2(1-(p/n)(2-\epsilon)) < 1$. This leads to

$$\| \, | u_{x_i} |^{\sigma_1} | u_t |^{\sigma_2} \|_{0,p} \leqslant C_{20} \| u \|_{1,2}^{\sigma_1+\sigma_2-1} \| u \|_{2-\epsilon,p}^{1-\sigma_2} \| u_t \|_{2-\epsilon,p}^{\sigma_2}.$$

(d) The last term coming from the nonlinearity f is $\| \, | u_{x_i} |^{\sigma_1} | u_{tx_j} |^{\sigma_3} \|_{0,p}$.

(α) If $\sigma_3 \geqslant 1$, we have

$$\| \, | u_{x_i} |^{\sigma_1} | u_{tx_j} |^{\sigma_3} \|_{0,p} \leqslant C_{21}(\| u_{x_i} \|_{0,p\tilde{p}\sigma_1}^{\sigma_1} \| u_{tx_j} \|_{0,p\tilde{q}} + \| \, | u_{x_i} |^{\sigma_1} | u_{tx_j} |^{\bar{\bar{\sigma}}_3} \|_{0,p}) \tag{9}$$

where $\sigma_1/2 + \bar{\bar{\sigma}}_3(\frac{1}{2} + 1/n) = 2/n + \frac{1}{2} - \delta$, $\delta > 0$ sufficiently small. The first term of (9) is treated for $p = n/2$ or $p = \min(2, n/2)$ with $1/\tilde{p} := p\sigma_1/2 < \frac{1}{2}$ and $1/\tilde{q} := 1 - 1/\tilde{p}$. We have $1/p\tilde{q} = 1/p - \sigma_1/2 > 1/p - (1-\epsilon)/n$, and therefore

$$\| u_{x_i} \|_{0,p\tilde{p}\sigma_1}^{\sigma_1} \ u_{tx_j} \|_{0,p\tilde{q}} \leqslant C_{22} \| u \|_{1,2}^{\sigma_1} \| u_t \|_{2-\epsilon,p}.$$

The second term in (9) is treated for $p = \min(2, n/2)$ by estimating

$$\| \, | u_{x_i} |^{\sigma_1} | u_{tx_j} |^{\bar{\bar{\sigma}}_3} \|_{0,p} \leqslant \| u_{x_i} \|_{0,p\tilde{p}\sigma_1}^{\sigma_1} \| u_{tx_j} \|_{0,p\tilde{q}\bar{\bar{\sigma}}_3}^{\bar{\bar{\sigma}}_3}.$$

Here we define $1/\tilde{p} := p\sigma_1/2$ and $1/\tilde{q} := 1 - 1/\tilde{p}$ so that an application of the growth restrictions gives $1/p\tilde{q}\bar{\bar{\sigma}}_3 < \frac{1}{2}$ which allows the use of Lemma 2.2 with $a = 1/\bar{\bar{\sigma}}_3 \cdot (1/p - \frac{1}{2} - 2/n - \delta)/(1/p - \frac{1}{2} - 2/n + \epsilon/n) < 1/\bar{\bar{\sigma}}_3$. This leads to

$$\| \, | u_{x_i} |^{\sigma_1} | u_{tx_j} |^{\sigma_3} \|_{0,p} \leqslant C_{23} \| u \|_{1,2}^{\sigma_1} (\| u_t \|_{2-\epsilon,p} + 1)(\| u_t \|_{0,2}^{\bar{\bar{\sigma}}_3(1-a)} + 1).$$

(β) The case $\sigma_3 < 1$ is treated for $p = n/2$ or $p = \min(2, n/2)$. We apply Hölder's inequality to the factors $| u_{x_i} |^{\sigma_1+\sigma_3-1}$, $| u_{x_i} |^{1-\sigma_3}$, and $| u_{tx_j} |^{\sigma_3}$ with corresponding exponents $\tilde{p}$, $\tilde{q}$, $\tilde{r}$ restricted by $1/\tilde{p} := p(\sigma_1 + \sigma_3 - 1)/2 < \frac{1}{2}$, $1/p \geqslant 1/p\tilde{q}(1-\sigma_3) \geqslant 1/p - (1-\epsilon)/n$, $1/p \geqslant 1/p\tilde{r}\sigma_3 \geqslant 1/p - (1-\epsilon)/n$. We remark that by use of the growth conditions $1/\tilde{p} + (1-\sigma_3)(1/p - (1-\epsilon)/n) + \sigma_3 p(1/p - (1-\epsilon)/n) < 1$ so that this choice is possible. This gives

$$\| \, | u_{x_i} |^{\sigma_1} | u_{tx_j} |^{\sigma_3} \|_{0,p} \leqslant C_{24} \| u \|_{1,2}^{\sigma_1+\sigma_3-1} \| u \|_{2-\epsilon,p}^{1-\sigma_3} \| u_t \|_{2-\epsilon,p}^{\sigma_3}.$$

(e) If $n \leqslant 3$ there might also appear a term of the form $\| g_i'(u)\, u_{x_i} u_t \|_{0,p}$, which can be estimated as follows:

$$| g_i'(u)\, u_{x_i} u_t | \leqslant C_{25}(| u |^{\rho_0^{(i)}} + 1)\, | u_{x_i} |\, | u_t |$$

$$\leqslant C_{26}(| u |^{\rho_0^{(i)} p_0^{(i)}}\, | u_t | + | u_{x_i} |^{p_1^{(i)}}\, | u_t | + | u_{x_i} |\, | u_t |).$$

Here we set $1/p_1^{(i)} := (n + \epsilon)/4 < 1$ and $1/p_0^{(i)} := 1 - 1/p_1^{(i)}$. An easy calculation shows $\rho_0^{(i)} p_0^{(i)}(\frac{1}{2} - 1/n) < 2/n = 1/p$, and $p_1^{(i)} + 1 < (n + 4)/n$ so that parts (a) and (c) of this proof directly lead to

$$\| g_i'(u)\, u_{x_i} u_t \|_{0,p} \leqslant C_{27}(\| u \|_{1,2}^{\rho_0^{(i)} p_0^{(i)}} + \| u \|_{1,2}^{p_1^{(i)}} + \| u \|_{1,2})\, \| u_t \|_{2-\epsilon,p},$$

if $p = n/2$.

Finally we have $\| g_i(u)\, u_{x_i t} \|_{0,p} \leqslant C_{28}(1 + \| u \|_{1,2}^{\rho_0^{(i)}+1})\, \| u_t \|_{2-\epsilon,p}$. This can be easily proved using part (b), because

$$(\rho_0^{(i)} + 1)\left(\frac{1}{2} - \frac{1}{n}\right) + \frac{1}{2} + \frac{1}{n} < \frac{1}{2} + \frac{2}{n}.$$

Summarizing the results of parts (a)–(e) we can conclude from (7) by Gronwall's lemma and Lemma 2.1 that an a-priori-estimate of $\| u_t(t) \|_{2-\epsilon,p}$ for $p = \min(2, n/2)$ is established.

In a next step we want to show an analogous result for $p = n/2$. An analysis of the proof up to this point easily shows that to this end only two terms have to be estimated in the case $n \geqslant 4$. The first of these is $\| \, | u |^{\sigma_0}\, | u_{tx_i} |^{\bar{\sigma}_3} \|_{0,p}$ where $\bar{\sigma}_3$ is defined as above, $\bar{\sigma}_3 \geqslant 1$. We have

$$\| \, | u |^{\sigma_0}\, | u_{tx_i} |^{\bar{\sigma}_3} \|_{0,p} \leqslant \| u \|_{0,p\tilde{p}\sigma_0}^{\sigma_0}\, \| u_{tx_i} \|_{0,p\tilde{q}(\bar{\sigma}_3-1)}^{\bar{\sigma}_3}\, \| u_{tx_i} \|_{0,p\tilde{r}},$$

where $\tilde{p}$, $\tilde{q}$, $\tilde{r}$ are restricted as follows:

$$1 = \frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} + \frac{1}{\tilde{r}}, \qquad 1 \geqslant \frac{1}{\tilde{r}} \geqslant p\left(\frac{1}{p} - \frac{1-\epsilon}{n}\right) = \frac{1+\epsilon}{2},$$

$$\frac{p}{2}(\bar{\sigma}_3 - 1) \geqslant \frac{1}{\tilde{q}} \geqslant p(\bar{\sigma}_3 - 1)\left(\frac{1}{2} - \frac{1-\epsilon}{n}\right),$$

$$\frac{p}{2}\sigma_0 \geqslant \frac{1}{\tilde{p}} \geqslant p\sigma_0\left(\frac{1}{2} - \frac{2-\epsilon}{n}\right).$$

This choice is possible because a calculation involving the definition of $\bar{\sigma}_3$ shows:

$$\frac{1+\epsilon}{2} + p(\bar{\sigma}_3 - 1)\left(\frac{1}{2} - \frac{1-\epsilon}{n}\right) + p\sigma_0\left(\frac{1}{2} - \frac{2-\epsilon}{n}\right) < 1.$$

Sobolev's lemma then gives

$$\| \, |u|^{\sigma_0} \, |u_{tx_i}|^{\bar{\sigma}_3} \|_{0,p} \leqslant C_{29} \| u \|_{2-\epsilon,2}^{\sigma_0} \| u_t \|_{2-\epsilon,2}^{\bar{\sigma}_3 - 1} \| u_t \|_{2-\epsilon,p}$$

The second term which has to be treated is $\| \, |u_{x_i}|^{\sigma_1} \, |u_{tx_j}|^{\bar{\bar{\sigma}}_3} \|_{0,p}$ with $\bar{\bar{\sigma}}_3$ as above, $\bar{\bar{\sigma}}_3 \geqslant 1$. We apply Hölder's inequality to the factors $|u_{x_i}|^{\sigma_1}$, $|u_{tx_j}|^{\bar{\bar{\sigma}}_3 - 1}$, and $|u_{tx_j}|$ with exponents $\tilde{p}$, $\tilde{q}$, $\tilde{r}$ which fulfill:

$$\frac{p\sigma_1}{2} \geqslant \frac{1}{\tilde{p}} \geqslant p\sigma_1 \left(\frac{1}{2} - \frac{1-\epsilon}{n} \right),$$

$$\frac{p(\bar{\bar{\sigma}}_3 - 1)}{2} \geqslant \frac{1}{\tilde{q}} \geqslant p(\bar{\bar{\sigma}}_3 - 1) \left(\frac{1}{2} - \frac{1-\epsilon}{n} \right), \qquad 1 \geqslant \frac{1}{\tilde{r}} \geqslant p \left(\frac{1}{p} - \frac{1-\epsilon}{n} \right).$$

Using the definition of $\bar{\bar{\sigma}}_3$ one can calculate that the sum of the right hand sides of these three inequalities is less than 1, so that they can be fulfilled. We get

$$\| \, |u_{x_i}|^{\sigma_1} \, |u_{tx_j}|^{\bar{\bar{\sigma}}_3} \|_{0,p} \leqslant C_{30} \| u \|_{2-\epsilon,2}^{\sigma_1} \| u_t \|_{2-\epsilon,2}^{\bar{\bar{\sigma}}_3 - 1} \| u_t \|_{2-\epsilon,p} .$$

We then go again back to (7), use the estimate of $\| u_t(t) \|_{2-\epsilon,p}$ given above, and arrive by Gronwall's lemma at the claimed result.

LEMMA 2.4. *Under the assumptions of the preceding lemma the following estimate holds:*

$$\| u_t(t) \|_{2-\epsilon,\hat{p}} \leqslant \omega_3(t) \qquad \textit{with} \qquad \hat{p} = n \quad \textit{for any } \epsilon > 0.$$

Proof. We proceed as in the foregoing proof. For part (a) we only remark that $H^{2-\epsilon,n/2} \subset L^p$ for any fixed p, if ϵ is chosen small enough, so that we arrive at $\| \, |u|^{\sigma_0} \, |u_t|^{\sigma_2} \|_{0,\hat{p}} \leqslant \omega_4(t) \, (\| u \|_{2-\epsilon,\hat{p}} + \| u_t \|_{2-\epsilon,\hat{p}})$ by Lemma 2.3. For part (b)(α) we apply Hölder's inequality to the factors $|u|^{\sigma_0}$, $|u_{tx_i}|^{\sigma_3 - 1}$, $|u_{tx_i}|$ with exponents $\tilde{p}$, $\tilde{q}$, $\tilde{r}$ subject to the conditions $\tilde{p} := 1/\epsilon\sigma_0$, i.e. $1/\tilde{p}\hat{p}\sigma_0 = \epsilon/n = 2/n - (2-\epsilon)/n$, $1/\tilde{r} \geqslant \epsilon$, i.e. $1/\tilde{r}\hat{p} \geqslant \epsilon/n = 1/\hat{p} - (1-\epsilon)/n$, and $1/\tilde{q} \geqslant (1+\epsilon)(\sigma_3 - 1)$, i.e. $1/\hat{p}\tilde{q}(\sigma_3 - 1) \geqslant (1+\epsilon)/\hat{p} = 2/n - (1-\epsilon)/n$. These conditions can be fulfilled because $\sigma_3 < 2$, so that we have

$$\| \, |u|^{\sigma_0} \, |u_{tx_i}|^{\sigma_3} \|_{0,\hat{p}} \leqslant C_{31} \| u \|_{2-\epsilon,n/2}^{\sigma_0} \| u_t \|_{2-\epsilon,n/2}^{\sigma_3 - 1} \| u_t \|_{2-\epsilon,\hat{p}} .$$

Part (b)(β) is treated by using the imbedding $H^{2-\epsilon,n/2} \subset L^p$ for any fixed $p < \infty$. For the next term (part c) we consider first of all the case $\sigma_1 \geqslant 1$ and conclude:

$$\| \, |u_{x_i}|^{\sigma_1} \, |u_t|^{\sigma_2} \|_{0,\hat{p}} \leqslant \| u_{x_i} \|_{0,\hat{p}\tilde{p}(\sigma_1 - 1)}^{\sigma_1 - 1} \| u_{x_i} \|_{0,\hat{p}\tilde{q}} \| u_t \|_{0,\hat{p}\tilde{r}\sigma_2}^{\sigma_2} ,$$

setting $1/\tilde{p} := (1+\epsilon)(\sigma_1 - 1) < 1$ (for we have $\sigma_1 < 2$ in any case), i.e. $1/\hat{p}\tilde{p}(\sigma_1 - 1) = (1+\epsilon)/n = 2/n - (1-\epsilon)/n$, $1/\tilde{r} := \epsilon\sigma_2$, i.e. $1/\hat{p}\tilde{r}\sigma_2 = \epsilon/n =$

$2/n - (2 - \epsilon)/n$, and $1/\tilde{q} := 1 - 1/\tilde{p} + 1/\tilde{r} > 0$. From this it follows that $1/\hat{p}\tilde{q} = 2/n - \sigma_1/n - \epsilon(\sigma_1 + \sigma_2 - 1) \geqslant \epsilon/n = 1/\hat{p} - (1 - \epsilon)/n$, so that we are led to

$$\| \, | u_{x_i} |^{\sigma_1} \, | u_t |^{\sigma_2} \|_{0,\hat{p}} \leqslant C_{32} \| u \|_{2-\epsilon,n/2}^{\sigma_1 - 1} \| u_t \|_{2-\epsilon,n/2}^{\sigma_2} \| u \|_{2-\epsilon,\hat{p}} .$$

The case $\sigma_1 < 1$ can be treated similarly leading to

$$\| \, | u_{x_i} |^{\sigma_1} \, | u_t |^{\sigma_2} \|_{0,\hat{p}} \leqslant C_{33} \| u_t \|_{2-\epsilon,n/2}^{\sigma_2} (\| u \|_{2-\epsilon,\hat{p}} + 1).$$

Next we treat part (d)(α) by using Hölder's inequality:

$$\| \, | u_{x_i} |^{\sigma_1} \, | u_{tx_j} |^{\sigma_3} \|_{0,\hat{p}} \leqslant \| u_{x_i} \|_{0,\tilde{p}\hat{p}\sigma_1}^{\sigma_1} \| u_{tx_j} \|_{0,\tilde{q}\hat{p}(\sigma_3 - 1)}^{\sigma_3 - 1} \| u_{tx_j} \|_{0,\tilde{r}\hat{p}} .$$

We define $1/\tilde{r} := \epsilon$, which means $1/\tilde{r}\hat{p} = \epsilon/n = 1/\hat{p} - (1 - \epsilon)/n$, and let $\tilde{p}$, $\tilde{q}$ fulfill: $2\sigma_1 \geqslant 1/\tilde{p} \geqslant \sigma_1(1 + \epsilon)$, i.e. $2/n \geqslant 1/\tilde{p}\hat{p}\sigma_1 \geqslant 2/n - (1 - \epsilon)/n$ (remark: $\sigma_1 < 1$), $1/\tilde{q} := 1 - 1/\tilde{r} - 1/\tilde{p} \geqslant 1 - \epsilon - 1/\tilde{p} > 0$. Then we calculate by the growth restrictions: $1/\tilde{q}\hat{p}(\sigma_3 - 1) > 2/n - (1 - \epsilon)/n$. Thus we have

$$\| \, | u_{x_i} |^{\sigma_1} \, | u_{tx_j} |^{\sigma_3} \|_{0,\hat{p}} \leqslant C_{34} \| u \|_{2-\epsilon,n/2}^{\sigma_1} \| u_t \|_{2-\epsilon,n/2}^{\sigma_3 - 1} \| u_t \|_{2-\epsilon,\hat{p}} .$$

Part (d)(β) is treated similarly, and part (e) as follows:

$$\| \, | u |^{\rho_0^{(i)}} \, u_{x_i} u_t \|_{0,\hat{p}} \leqslant \| u \|_{0,\tilde{p}\hat{p}\rho_0^{(i)}}^{\rho_0^{(i)}} \| u_{x_i} \|_{0,\tilde{q}\hat{p}} \| u_t \|_{0,\tilde{r}\hat{p}}$$

with $1/\tilde{p} := \epsilon\rho_0^{(i)}$, $1/\tilde{q} \geqslant \epsilon$, $1/\tilde{r} \geqslant \epsilon$, $1 = 1/\tilde{p} + 1/\tilde{q} + 1/\tilde{r}$. Sobolev's imbedding theorem gives:

$$\| g_i'(u) \, u_{x_i} u_t \|_{0,\hat{p}} \leqslant C_{35} (\| u \|_{2-\epsilon,n/2}^{\rho_0^{(i)}} + 1) \| u_t \|_{2-\epsilon,n/2} \| u \|_{2-\epsilon,\hat{p}} .$$

Part (f) can be treated easily, too, so that the claimed estimate follows as in the preceding lemma.

LEMMA 2.5. *Under the assumptions of Lemma 2.3 the following estimate holds:*

$$\| u_t(t) \|_{l+1-\epsilon,p} \leqslant \omega_4(t) \qquad \textit{for any } \epsilon > 0, \;\; \frac{n}{2} \leqslant p < \infty,$$

where $l = [n/2] + 2$.

Proof. From Lemma 2.4 we have $\| u_t(t) \|_{1,p} \leqslant \omega_5(t)$ for each $n/2 \leqslant p < \infty$, so that an a-priori-estimate of $\| u_t(t) \|_{2-\epsilon,p}$, $n/2 \leqslant p < \infty$, can be given analogously to the preceding lemma. In order to estimate $\| u_t(t) \|_{\bar{l}-\epsilon,p}$, starting from a given estimate of $\| u_t(t) \|_{\bar{l}-1-\epsilon,\tilde{p}}$ ($\bar{l} \geqslant 3$, $\bar{l} \in \mathbb{N}$) for $n/2 \leqslant \tilde{p} < \infty$, we remark that the following inequality can easily be proved with continuous functions F_α and

$$v \in \{u, u_t\}: \qquad \| F(u, \nabla u, u_t, \nabla u_t)\|_{\tilde{l}-2,p}$$

$$\leqslant C_{36} \Bigg(\sum_{\substack{1 \leqslant |\alpha_i| \leqslant \tilde{l}-2 \\ \tilde{l}-2 \geqslant j \geqslant 1}} \| F_\alpha(u, \nabla u, u_t, \nabla u_t)\, D^{\alpha_1} v \cdots D^{\alpha_j} v$$

$$\cdot (1 + | D^{\tilde{l}-1} u | + | D^{\tilde{l}-1} u_t |)\|_{0,p} + \| F(u, \nabla u, u_t, \nabla u_t)\|_{0,p} \Bigg)$$

$$\leqslant \omega(\| u \|_{\tilde{l}-2,\infty}, \| u_t \|_{\tilde{l}-2,\infty}) (\| u \|_{\tilde{l}-1,p} + \| u_t \|_{\tilde{l}-1,p})$$

where $\omega \in C^0_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$. Because $\| u \|_{\tilde{l}-2,\infty}$ and $\| u_t \|_{\tilde{l}-2,\infty}$ were assumed to be already estimated, the inequality analogous to (7) gives an a-priori-bound of $\| u_t(t)\|_{\tilde{l}-\epsilon,p}$, so that the lemma is proved.

Remark. We always assumed $K \geqslant n/2 + [n/2] + 3 = n/2 + l + 1$, so that $H^{l+1-\epsilon,p} \supset H^{K,2}$, and therefore it makes sense to estimate the $H^{l+1-\epsilon,p}$ norm a-priori for the given local solution.

LEMMA 2.6. *Under the assumptions of Lemma* 2.3 *we also have* $\| u_t(t)\|_{K,2} \leqslant \omega_6(t)$.

Proof. Let l be as in Lemma 2.5. Then one has as in the preceding lemma $\| F(u, \nabla u, u_t, \nabla u_t)\|_{l-1,2} \leqslant \omega(\| u \|_{l-1,\infty}, \| u_t \|_{l-1,\infty})\ (\| u \|_{l,2} + \| u_t \|_{l,2})$ with a function $\omega \in C^0_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, so that an estimate of $\| u_t(t)\|_{l,2}$ follows.

Assume an estimate of $\| u_t(t)\|_{\tilde{K},2}$ for a fixed $l \leqslant \tilde{K} < K$, $\tilde{K} \in \mathbb{N}$. Because the chain rule together with Sobolev's imbedding theorem gives

$$\| F(u, \nabla u, u_t, \nabla u_t)\|_{\tilde{K},2} \leqslant \omega(\| u \|_{\tilde{K},2}, \| u_t \|_{\tilde{K},2}) (\| u \|_{\tilde{K}+1,2} + \| u_t \|_{\tilde{K}+1,2})$$

with ω as above (remark $\tilde{K} - 1 > n/2$), an application of Gronwall's lemma to an analogue of (7) gives an estimate of $\| u_t(t)\|_{\tilde{K}+1,2}$, and so step-by-step the desired result.

THEOREM 2.1. *Under the assumption of Lemma* 2.3 *the Cauchy-problem* (4) *has a unique solution u in the class* $C^1([0, T], H^{K,2}(\mathbb{R}^n)) \cap C^2([0, T], H^{K-2,2}(\mathbb{R}^n))$ *for any* $T > 0$. *This solution is a classical one.*

Proof. The remarks at the beginning of this paragraph together with Lemma 2.6 directly give the result.

Now we want to specialize our results to the equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta \frac{\partial u}{\partial t} + f(u) = 0, \qquad u(x, 0) = \varphi(x), \qquad \frac{\partial u}{\partial t}(x, 0) = \psi(x), \qquad (10)$$

where f fulfills the following conditions:

$$f \in C^{\infty}_{\text{loc}}(\mathbb{R}, \mathbb{R}), \qquad f(0) = 0, \qquad f'(S) \geqslant 0 \quad \text{for all } S \in \mathbb{R}. \tag{11}$$

In this case we shall show that some better results hold. Lemma 2.1 gives first of all an a-priori-bound of $\| u_t(t)\|^2 + \int_0^t \| u_t(S)\|_{1,2}^2 \, dS$. Moreover we have

LEMMA 2.7. $\| u(t)\|_{2,2} \leqslant \omega_7(t)$.

Proof. Multiply the differential equation by Au, $A := -\Delta + 1$. This gives

$$\frac{d}{dt}(u_t, Au) - \| A^{1/2} u_t \|^2 + \frac{1}{2}\frac{d}{dt}\| Au \|^2 - \frac{1}{2}\frac{d}{dt}\| A^{1/2} u \|^2 + (f(u), Au) = 0.$$

Now we have

$$(f(u), Au)$$
$$= -\sum_{i=1}^{n} \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i}(f(u)\, u_{x_i})\, dx + \sum_{i=1}^{n} \int_{\mathbb{R}^n} f'(u)\, u_{x_i}^2 \, dx + \int_{\mathbb{R}^n} f(u)\, u \, dx \geqslant 0$$

so that by integration we are led to

$$\tfrac{1}{2}\| Au(t)\|^2 + (u_t(t), Au(t))$$
$$\leqslant \tfrac{1}{2}\| A\varphi \|^2 + (\psi, A\varphi) + \| \tfrac{1}{2} A^{1/2} u(t)\|^2 - \tfrac{1}{2}\| A^{1/2}\varphi \|^2 + \int_0^t \| A^{1/2} u_t(S)\|^2 \, dS,$$

which implies

$$\tfrac{1}{2}\| Au(t)\|^2 \leqslant \tfrac{1}{4}\| Au(t)\|^2 + \| u_t(t)\|^2 + \tfrac{1}{2}\| A\varphi \|^2 + \| \psi \| \, \| A\varphi \| + \tfrac{1}{2}\| A^{1/2} u(t)\|^2$$
$$- \tfrac{1}{2}\| A^{1/2}\varphi \|^2 + \int_0^t \| A^{1/2} u_t(S)\|^2 \, dS.$$

Using the a-priori-bounds above the result follows directly.

LEMMA 2.8. *Let f fulfill the following growth condition* $| f(S)| \leqslant C_{37}(| S |^{\rho_0} + S)$ *with* $\rho_0 = n/(n-4) - \delta$, $\delta > 0$ *arbitrarily small, for* $n > 4$, $\rho_0 < \infty$ *for* $n = 4$. *Then we have* $\| u_t(t)\|_{2-\epsilon,p} \leqslant \omega_8(t)$ *for any* $\epsilon > 0$, $p = n/2$.

Proof. Hölder's inequality gives $\| \, | u |^{\rho_0} \|_{0,p} \leqslant \| u \|_{0,p\tilde{p}(\rho_0-1)}^{\rho_0-1} \| u \|_{0,p\tilde{q}}$. Setting $1/\tilde{q} := \epsilon$, which implies $1/p\tilde{q} = \epsilon/n = 1/p - (2-\epsilon)/n$ and $1/\tilde{p} := 1 - \epsilon$, so that $1/p\tilde{p}(\rho_0 - 1) > \frac{1}{2} - 2/n$ for $n \geqslant 4$, we arrive at $\| f(u)\|_{0,p} \leqslant C_{38}(\| u \|_{2,2}^{\rho_0-1} + 1) \| u \|_{2-\epsilon,p}$. The result now follows as above.

THEOREM 2.2. *Assume f fulfills* (11) *and the growth condition of the preceding lemma,* $\varphi, \psi \in H^{K,2}(\mathbb{R}^n)$, $K \geqslant n/2 + [n/2] + 3$. *Then problem* (10) *has a unique (classical) solution*

$$u \in C^1([0, T], H^{K,2}(\mathbb{R}^n)) \cap C^2([0, T], H^{K-2,2}(\mathbb{R}^n)) \qquad \textit{for any } T > 0.$$

Proof. Cf. Lemma 2.4ff.

EXAMPLES. 1. An application of Theorem 2.1 shows the existence of global classical solutions of the Cauchy-problem for equations of the type

$$\frac{\partial^2 u}{\partial t^2} - \Delta \frac{\partial u}{\partial t} + C_1 u^2 u_t + C_2 u^3 + C_3 \sum_{i=1}^{3} \left(\frac{\partial u}{\partial x_i} \frac{\partial u}{\partial t} + 2u \frac{\partial^2 u}{\partial t\, \partial x_i} \right) = 0 \quad \text{for } n = 3$$

where C_1, C_2, $C_3 \geqslant 0$ (cf. the paper of Arima–Hasegawa [1] for the special case $C_3 = 0$).

In $n = 2$ dimensions we can treat e.g. equations of the form

$$\frac{\partial^2 u}{\partial t^2} - \Delta \frac{\partial u}{\partial t} + C_1 u^{2p} u_t + C_2 u^{2r+1} + C_3 \sum_{i=1}^{2} \left(u^{q_i} u_{x_i} u_t + \frac{2}{q_i + 1} u^{q_i+1} u_{tx_i} \right) = 0.$$

where p, r, $q_i \in \mathbb{N}$; C_1, C_2, $C_3 \geqslant 0$.

For $n = 1$ very general equations can be treated, such as

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^3 u}{\partial t\, \partial x^2} + C_1 u^{2p} u_t^3 + C_2 u_x^2 u_t + C_3 u^{2q} u_x^2 u_t + C_4 u^{2r+1}$$
$$+ C_5 \left(u^S u_x u_t + \frac{2}{S+1} u^{S+1} u_{tx} \right) = 0,$$

where p, q, r, $S \in \mathbb{N}$; C_1, C_2, C_3, C_4, $C_5 \geqslant 0$.

More general examples are for arbitrary dimension $n \geqslant 3$:

$$\frac{\partial^2 u}{\partial t^2} - \Delta \frac{\partial u}{\partial t} + (C_1 g_1(u) + C_2 g_2(\nabla u))\, u_t + C_3 g_3(u) = 0,$$

where

$$C_1, C_2, C_3 \geqslant 0, \qquad g_1(u) = |u|^{\sigma_0} \quad \text{for } |u| \geqslant 1,$$
$$\sigma_0 < \frac{4}{n-2}, \qquad g_1 \in C^\infty_{\text{loc}}(\mathbb{R}), \qquad g_2(\nabla u) = |\nabla u|^{\sigma_1} \quad \text{for } |\nabla u| \geqslant 1,$$
$$\sigma_1 < \frac{4}{n}, \qquad g_2 \in C^\infty_{\text{loc}}(\mathbb{R}^n), \qquad \text{and} \qquad g_3(u) = |u|^\rho u \quad \text{for } |u| \geqslant 1,$$
$$\rho < \frac{4}{n-2}, \qquad g_3 \in C^\infty_{\text{loc}}(\mathbb{R}).$$

2. Examples for using Theorem 2.2 are:

$$\frac{\partial^2 u}{\partial t^2} - \Delta \frac{\partial u}{\partial t} + u^{2p+1} = 0, \qquad p \in \mathbb{N}, \quad n \leqslant 4$$

$$\frac{\partial^2 u}{\partial t^2} - \Delta \frac{\partial u}{\partial t} + u^3 = 0, \qquad \text{for } n = 5.$$

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